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Bi-conformal vector fields and the local geometric characterization of conformally separable pseudo-Riemannian manifolds II

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Abstract

In this paper we continue the study of bi-conformal vector fields started in [A. García-Parrado, J.M.M. Senovilla, Class. Quant. Grav. 21 (2004) 2153–2177]. These are vector fields defined on a pseudo-Riemannian manifold by the differential conditions $\pounds_{\vec{\xi}}\, P_{ab} = \phi P_{ab}$, $\pounds_{\vec{\xi}}\, \Pi_{ab} = \chi \Pi_{ab}$, where P_{ab}, Π_{ab} are orthogonal and complementary projectors with respect to the metric tensor g_{ab}. In a previous paper we explained how the analysis of these differential conditions enabled us to derive local geometric characterizations of the most relevant cases of *conformally separable* (also called double twisted) pseudo-Riemannian manifolds. In this paper we carry on this analysis further and provide local invariant characterizations of conformally separable pseudo-Riemannian manifolds with *conformally flat* leaf metrics. These characterizations are rather similar to that existing for conformally flat pseudo-Riemannian manifolds but instead of the Weyl tensor, we must demand the vanishing of certain four rank tensors constructed from the curvature of an affine, non-metric, connection (bi-conformal connection). We also speculate with possible applications to finding results for the existence of foliations by conformally flat hypersurfaces in any pseudo-Riemannian manifold.

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1. Introduction

One of the most interesting problems faced in Differential Geometry is the invariant characterization of pseudo-Riemannian manifolds. The most famous exposition about this issue was stated

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by Klein in his *Erlangen program* [11] where he set about to the ambitious task of classifying all the possible geometries be they underlay by a pseudo-Riemannian manifold or not.

If we stick to pseudo-Riemannian manifolds then the invariant characterizations are usually achieved by means of the definition of intrinsic (coordinate independent) geometric objects upon which a condition is imposed. The best known examples are flat and conformally flat pseudo-Riemannian manifolds which are those such that the Riemann tensor and the Weyl tensor are zero, respectively. Other relevant cases found in the framework of General Relativity are the Schwarzschild geometry [6], Kerr black hole [16,12], plane fronted waves with parallel rays and many others which are invariantly characterized by certain geometric conditions.

Symmetry considerations often play an important role in invariant characterizations (in fact they lie at the heart of the Erlangen program). They come in the form of concrete Lie groups acting on the manifold under study (group realizations) and by knowing the orbits or the isotropy subgroups of such actions it is possible sometimes to identify the pseudo-Riemannian manifold.

The above description of symmetries is performed in terms of *finite groups* but we can also study the *infinitesimal generators* of these symmetry groups. These are vector fields satisfying certain differential conditions which in general involve the Lie derivative. For instance in the case of isometries and conformal transformations the corresponding differential conditions fulfilled by the infinitesimal generators are

$$\pounds_{\vec{\xi}}\, g_{ab} = 0 \quad \text{(isometries)}, \qquad \pounds_{\vec{\xi}}\, g_{ab} = 2\phi g_{ab} \quad \text{(conformal motions)}, \tag{1}$$

where ϕ is a smooth function. There are other examples of symmetries studied in the literature (see e.g. [10,7,17]) all of them involving the basic objects in Differential Geometry (Levi–Civita connection, Riemann curvature, Ricci tensor, etc.). However, not many examples have been tackled with more general differential conditions (some of them can be found in [9,1,18]). In any case from the study of the differential conditions satisfied by the vector fields generating the symmetry we can sometimes get the geometric conditions fulfilled by the pseudo-Riemannian manifolds admitting the symmetry under study and these conditions are precisely the geometric characterizations alluded to above.

In all the cases we are aware of, the differential conditions are linear in the generating vector fields and one can show that these vector fields are (local) Lie algebras. If such Lie algebras are finite-dimensional then it is possible to derive from the differential conditions the *normal form* which generically looks like

$$D_{x^a}\Phi_B = f(x^1, \ldots, x^n, \Phi^1, \ldots, \Phi^m),$$

where $\{x^1, \ldots, x^n\}$ are local coordinates, $D_{x^a}, a = 1, \ldots n$ a differential operator and $\Phi^1, \ldots, \Phi^m$ a set of variables called the *system variables*. Particularly interesting is the study of the *first integrability* and *complete integrability* conditions associated to the normal form because they are the geometric conditions under which a pseudo-Riemannian manifold admits locally a finite-dimensional Lie algebra of infinitesimal generators with the highest possible dimension. For instance a n-dimensional locally conformally flat pseudo-Riemannian manifold with a C^2 conformal factor can be characterized by the existence of a local Lie algebra of conformal motions of dimension $(n+1)(n+2)/2$ and the complete integrability conditions computed from (1) are just the vanishing of the Weyl tensor ($n > 3$) in a neighbourhood of a point. Therefore from the sole symmetry considerations one can obtain important geometric objects and this is one of our aims in this paper where we carry out the calculation of the first and the complete integrability conditions for the case of *bi-conformal vector fields* (see next).

In [3] a new symmetry transformation called bi-conformal transformation was put forward. The infinitesimal generators of such transformations are called bi-conformal vector fields and they are defined through the differential conditions

$$\pounds_{\vec{\xi}} P_{ab} = \phi P_{ab}, \qquad \pounds_{\vec{\xi}} \Pi_{ab} = \chi \Pi_{ab},$$

where P_{ab} and Π_{ab} are orthogonal and complementary projectors with respect to the metric g_{ab}. This study was improved in [2] where we obtained a simple expression of the normal form associated to the above conditions and obtained invariant, coordinate free, characterizations of the most relevant cases of *conformally separable* pseudo-Riemannian manifolds (we will quote this paper very often in this work and so it will be referred to as paper I). A pseudo-Riemannian manifold is conformally separable at a point q if there exist a local coordinate chart $x = \{x^1, \ldots, x^n\}$ based at q such that the metric tensor takes the form

$$\mathrm{d}s^2 = \Xi_1(x) G_{\alpha\beta}\, \mathrm{d}x^\alpha\, \mathrm{d}x^\beta + \Xi_2(x) G_{AB}\, \mathrm{d}x^A\, \mathrm{d}x^B, \quad 1 \le \alpha, \beta \le p,\ 1 \le A, B \le n - p, \tag{2}$$

where the functions $G_{\alpha\beta}$, G_{AB} only depend on the coordinates labelled by their respective subindexes (see Definition 10). The metrics $\Xi_1(x)G_{\alpha\beta}$, $\Xi_2(x)G_{AB}$ are called *leaf metrics* of the separation. If $G_{\alpha\beta}$, G_{AB} are flat metrics then the manifold is *bi-conformally flat* at q.

This paper is the continuation of the work started in paper I. There we introduced a new affine connection (bi-conformal connection) and explained how its use allows us to obtain relevant geometric information out of the study bi-conformal vector fields. In concrete terms we showed that a pseudo-Riemannian manifold is conformally separable at a point if and only if a certain rank 3 tensor T_{abc} constructed from a pair of orthogonal and complementary projectors is zero in a neighbourhood of that point. These projectors are naturally identified with the leaf metrics (see Section 5 for more details about this identification). In this work a local geometric characterization for bi-conformally flat spaces in terms of a certain tensor $T^a{}_{bcd}$ constructed also from the projectors P_{ab} and Π_{ab} is derived. We prove the remarkable result that both T_{abc}, $T^a{}_{bcd}$ are zero if and only if the space is bi-conformally flat at a point being P_{ab} and Π_{ab} the leaf metrics.[1] This is the translation to the bi-conformal case of the familiar condition that the Weyl tensor vanish for a pseudo-Riemannian manifold to be locally conformally flat. The aforementioned conditions will be encountered here as the complete integrability conditions associated to the differential conditions defining a bi-conformal vector field. We can also give geometric conditions under which a conformally separable pseudo-Riemannian manifold admits conformally flat leaf metrics. These conditions take the form

$$P^a{}_r P_b{}^s P_c{}^q P_d{}^t T^r{}_{sqt} = 0.$$

One of the most remarkable advantages of these characterizations is that they enable us to define intrinsically when a pseudo-Riemannian manifold is conformally separable and bi-conformally flat at a point. Furthermore we can test if two given orthogonal and complementary projectors P_{ab}, Π_{ab} give rise to a conformally separable pseudo-Riemannian manifold with conformally flat leaf metrics without actually finding the adapted local coordinates of (2).

The paper outline is as follows: in Section 2 we recall basic issues of paper I used in this work (the paper is self-contained and there is no need of reading paper I to understand the

[1] If any of the projectors have algebraic rank 3 then the conditions are slightly different.

results presented here). After this the paper is divided into two differenced parts: in the first one comprised by Sections 3 and 4 we have put all the issues dealing strictly with bi-conformal vector fields being these the first and the complete integrability conditions, respectively. In the second part we show how these conditions can be used to characterize locally bi-conformally flat pseudo-Riemannian manifolds and conformally separable pseudo-Riemannian manifolds with conformally flat foliations (Section 5). These results are gathered by Theorems 11 and 13 (the special case of any of the leaf metrics being of rank 3 is treated in Theorem 14). A reader only interested in these geometric characterizations should jump straight to this section and skip the long tensor calculations of Sections 3 and 4. Finally examples are provided in Section 6.

Part of the results presented in this paper relies on hefty tensor calculations. These calculations have been done by hand and double-checked with the newly released Mathematica package "xtensor" [13] with excellent agreement.

1.1. Notation conventions

The notation of the paper is standard. We work in a C^∞ connected pseudo-Riemannian manifold V with metric tensor g_{ab} and we use index notation for all objects constructed from the tensor bundles $T^r_s(V)$ of V. Square brackets enclosing indexes are used to denote antisymmetrization and whenever a set of indexes is between strokes it is excluded from the antisymmetrization operation. The metric tensor gives rise to the Levi–Civita connection $\gamma^a{}_{bc}$ (we reserve the nomenclature $\Gamma^a{}_{bc}$ for the connection components calculated in a natural basis) and the curvature tensor $R^a{}_{bcd}$, being our convention for the relation between these two

$$R^a{}_{bcd} \equiv \partial_c \Gamma^a{}_{db} - \partial_d \Gamma^a{}_{cb} + \Gamma^a{}_{rc}\Gamma^r{}_{db} - \Gamma^a{}_{rd}\Gamma^r{}_{cb}. \tag{3}$$

Under this convention the Ricci identity becomes

$$\nabla_b\nabla_c u^a - \nabla_c\nabla_b u^a = R^a{}_{rbc}u^r, \qquad \nabla_b\nabla_c u_a - \nabla_c\nabla_b u_a = -R^r{}_{abc}u_r,$$

where ∇_a is the covariant derivative of the Levi–Civita connection.

All the above relations are still valid if $\gamma^a{}_{bc}$ is a connection with no torsion (symmetric or affine connection).

The infinite-dimensional Lie algebra of smooth vector fields of the manifold V is denoted by $\mathfrak{X}(V)$. Finally the Lie derivative operator with respect to a vector field $\vec{\boldsymbol{\xi}}$ is $\pounds_{\vec{\xi}}$.

2. Preliminaries

In this section we review concepts of paper I which are needed in this work. We quote the results without proofs as they can all be found in paper I.

Definition 1. A smooth vector field $\vec{\boldsymbol{\xi}}$ on V is said to be a *bi-conformal vector field* if it fulfills the condition

$$\pounds_{\vec{\xi}}\, P_{ab} = \phi P_{ab}, \qquad \pounds_{\vec{\xi}}\, \Pi_{ab} = \chi \Pi_{ab}, \quad \phi, \chi \in C^\infty(V), \tag{4}$$

where P_{ab}, Π_{ab} are smooth sections of the tensor bundle $T^0_2(V)$ satisfying the properties

$$P_{ab} = P_{ba}, \qquad \Pi_{ab} = \Pi_{ba}, \qquad P_{ab} + \Pi_{ab} = g_{ab}, \qquad P_{ap}P^p{}_b = P_{ab},$$
$$\Pi_{ap}\Pi^p{}_b = \Pi_{ab}, \qquad P_{ap}\Pi^p{}_b = 0. \tag{5}$$

The geometrical meaning of these conditions is that P_{ab} and Π_{ab} are *orthogonal projectors* with respect to the metric tensor g_{ab} at each point of the manifold. We describe next briefly some useful properties of P_{ab} and Π_{ab} which will be used along the paper. To start with note that we can decompose the vector space $T_p(V)$ as a direct sum of the ranges of the endomorphisms $P^a{}_b$ and $\Pi^a{}_b$. These ranges are the respective eigenspaces of the endomorphisms with eigenvalue $+1$ and they are orthogonal subspaces. P_{ab} and Π_{ab} enable us to represent any *nondegenerate* smooth distribution D in a compact way.[2] To see this more clearly, let $\{u^a_1, \ldots, u^a_p\}$, $0 < p < n$, be an orthonormal set of smooth vector fields spanning the distribution D (such a set always exists if D is nondegenerate) with $\epsilon_\alpha = u^a_\alpha g_{ab} u^b_\alpha$. Then the smooth sections

$$P_{ab} \equiv \sum_{\alpha=1}^{p} \epsilon_\alpha u^\alpha_a u^\alpha_b, \qquad \Pi_{ab} \equiv g_{ab} - P_{ab}, \tag{6}$$

satisfy (5). Conversely, the smoothness of $P^a{}_b$ and $\Pi^a{}_b$ together with (5) guarantee that the ranges of $P^a{}_b$ and $\Pi^a{}_b$ span smooth distributions on V. To see this we need to show that the range dimension of each projector does not vary in the manifold V (these numbers are $p \equiv P^a{}_a$ and $n - p \equiv \Pi^a{}_a$) because in that case such ranges are smooth distributions (see e.g. [8]). Let us split[3] V in subsets A_k, $B_{k'} k, k' = 1, \ldots, n-1$ defined by the conditions

$$A_k = \{q \in V : P^a{}_a = k\}, \qquad B_{k'} = \{q \in V : \Pi^a{}_a = k'\},$$

and denote by $k_{\min}$ and $k_{\max}$, respectively, the minimum and maximum value of the integer k (similarly we define $k'_{\min}$, $k'_{\max}$). By the rank theorem (see e.g. Theorem 3.1 of [7]) $A_{k_{\max}}$ and $B_{k'_{\max}}$ are open sets and from the third property of (5) clearly $A_{k_{\max}} = B_{k'_{\min}}$, $B_{k'_{\max}} = A_{k_{\min}}$. On the other hand using again the rank theorem we deduce that $V \setminus B_{k'_{\min}}$, $V \setminus A_{k_{\min}}$ are open and thus $B_{k'_{\min}}$ and $A_{k_{\min}}$ must be closed from which we conclude that $A_{k_{\max}}$, $B_{k'_{\max}}$ are both open and closed at the same time and thus equal to V since it is connected.

The differential conditions (4) are the starting point for an interesting study of the properties and geometric significance of bi-conformal vector fields. In paper I we argued that the set of bi-conformal vector fields of a pseudo-Riemannian manifold V is a Lie subalgebra of $\mathfrak{X}(V)$ and we established the conditions under which such algebra is always finite-dimensional as well as its greatest dimension N. The Lie algebra is finite-dimensional if $p, n - p \neq 1, 2$ and in this case

$$N = \tfrac{1}{2}(p+1)(p+2) + \tfrac{1}{2}(n-p+1)(n-p+2).$$

The number N is calculated from the *normal form* associated with the differential conditions (4). This is a set of equations obtained from the differential conditions by means of successive differentiations and they were calculated in paper I (here, this normal form is recalled in (8)). In doing this calculation the introduction of a new affine connection (*bi-conformal connection*) revealed itself essential rendering the normal form very neatly. The components of the bi-conformal connection $\bar{\gamma}^a{}_{bc}$ are related to the Levi–Civita connection $\gamma^a{}_{bc}$ by the relation

$$\bar{\gamma}^a{}_{bc} = \gamma^a{}_{bc} + \frac{1}{2p}(E_b P^a{}_c + E_c P^a{}_b) + \frac{1}{2(n-p)}(W_b \Pi^a{}_c + W_c \Pi^a{}_b)$$
$$+ \frac{1}{2}(P^a{}_p - \Pi^a{}_p) M^p{}_{bc},$$

[2] By nondegenerate we mean that the scalar product g_{ab} restricted to the subspace of $T_p(V)$ generated by the distribution D is not degenerated.

[3] We are indebted to Graham Hall for this proof.

where

$$M_{abc} \equiv \nabla_b P_{ac} + \nabla_c P_{ab} - \nabla_a P_{bc}, \qquad E_a \equiv M_{abc} P^{bc}, \qquad W_a \equiv -M_{abc} \Pi^{bc}. \tag{7}$$

By definition the bi-conformal connection is an affine connection and we shall denote the covariant derivative and the curvature tensor of this connection by $\bar{\nabla}$ and $\bar{R}^a{}_{bcd}$, respectively. The bi-conformal connection does not stem from a metric tensor in general as will be shown in explicit examples and hence the tensor $\bar{R}^a{}_{bcd}$ does not fulfill the same properties as the curvature tensor of a metric connection. The Bianchi identities though, remain as in the case of a metric connection.

In paper I, we explained the role of the bi-conformal connection in the local geometric characterization of *conformally separable* pseudo-Riemannian manifolds (see Definition 10). This role will be strengthened in Section 5 where we will develop a local invariant characterization of conformally separable pseudo-Riemannian manifolds with conformally flat leaf metrics (*bi-conformally flat* pseudo-Riemannian manifolds).

3. First integrability conditions

As we commented before, one can differentiate Eq. (4) a certain number of times and then isolate the derivatives of certain variables (system variables) in terms of themselves thereby obtaining a "closed" or normal form. This calculation was accomplished in paper I and we reproduce next the result

$$\bar{\nabla}_a \phi = \bar{\phi}_a + \phi^*_a, \bar{\nabla}_a \chi = \bar{\chi}_a + \chi^*_a, \tag{8a}$$

$$\bar{\nabla}_b \phi^*_a = \frac{-1}{p}\left[\pounds_{\vec{\xi}} (\bar{\nabla}_b E_a) + \frac{1}{2}(\bar{\chi}_b E_a + \bar{\chi}_a E_b - (\bar{\chi}^r E_r)\Pi_{ab})\right], \tag{8b}$$

$$\bar{\nabla}_b \chi^*_a = \frac{1}{p-n}\left[\pounds_{\vec{\xi}} (\bar{\nabla}_b W_a) + \frac{1}{2}(\bar{\phi}_b W_a + \bar{\phi}_a W_b - (\bar{\phi}^r W_r)P_{ab})\right], \tag{8c}$$

$$\bar{\nabla}_b \bar{\phi}_c = \frac{1}{2-p}\left[\pounds_{\vec{\xi}} L^0_{bc} + 2\bar{\phi}^r \bar{\nabla}_r P_{bc}\right], \tag{8d}$$

$$\bar{\nabla}_b \bar{\chi}_c = \frac{1}{2-n+p}\left[\pounds_{\vec{\xi}} L^1_{bc} + 2\bar{\chi}^r \bar{\nabla}_r \Pi_{bc}\right], \tag{8e}$$

$$\bar{\nabla}_b \xi^a = \Psi_b{}^a, \tag{8f}$$

$$\bar{\nabla}_b \Psi_c{}^a = \frac{1}{2}(\bar{\phi}_b P^a{}_c + \bar{\phi}_c P^a{}_b - \bar{\phi}^a P_{cb} + \bar{\chi}_b \Pi^a{}_c + \bar{\chi}_c \Pi^a{}_b - \bar{\chi}^a \Pi_{cb}) - \xi^d \bar{R}^a{}_{cdb} \tag{8g}$$

with the definitions

$$\phi^*_a \equiv \Pi_{ab}\phi^b, \qquad \bar{\phi}_a \equiv P_{ab}\phi^b, \qquad \chi^*_a \equiv P_{ab}\chi^b, \qquad \bar{\chi}_a \equiv \Pi_{ab}\chi^b,$$

$$L^0{}_{bc} \equiv 2\left[P^d{}_r \bar{R}^r{}_{cdb} - \frac{1}{p}(P^d{}_c P^r{}_q \bar{R}^q{}_{rdb} + P^d{}_b P^r{}_q \bar{R}^q{}_{rdc} - P^r{}_q \bar{R}^q{}_{rbc})\right] + \frac{\bar{R}^0}{1-p} P_{bc},$$
$$\bar{R}^0 \equiv P^d{}_r \bar{R}^r{}_{cdb} P^{cb}, \tag{9}$$

$$L^1{}_{bc} \equiv 2\left[\Pi^d{}_r \bar{R}^r{}_{cdb} - \frac{1}{n-p}(\Pi^d{}_c \Pi^r{}_q \bar{R}^q{}_{rdb} + \Pi^d{}_b \Pi^r{}_q \bar{R}^q{}_{rdc} - \Pi^r{}_q \bar{R}^q{}_{rbc})\right]$$
$$+ \frac{\bar{R}^1}{1-n+p}\Pi_{bc}, \qquad \bar{R}^1 \equiv \Pi^d{}_r \bar{R}^r{}_{cdb} \Pi^{cb}. \tag{10}$$

The variables lying on the left-hand side of (8) are the system variables. Not all these variables are independent because as shown in paper I they are constrained by the conditions (constraint equations)

$$\pounds_{\vec{\xi}}\, P_{ab} = \phi P_{ab}, \qquad \pounds_{\vec{\xi}}\, \Pi_{ab} = \chi \Pi_{ab}, \tag{11a}$$

$$\pounds_{\vec{\xi}}\, E_a = -p\phi^*_a, \qquad \pounds_{\vec{\xi}}\, W_a = -(n-p)\chi^*_a. \tag{11b}$$

In this section we spell out the first integrability conditions of each equation of (8). These are geometric conditions arising from the compatibility conditions yielded by the commutation of two covariant derivatives. The commutation rules for these derivatives are given by the Ricci identity

$$\bar\nabla_a\bar\nabla_b \Xi^{a_1\cdots a_r}{}_{b_1\cdots b_s} - \bar\nabla_b\bar\nabla_a \Xi^{a_1\cdots a_r}{}_{b_1\cdots b_s} = \sum_{q=1}^{r} \bar R^{a_q}{}_{tab}\,\Xi^{a_1\cdots a_{q-1}ta_{q+1}\cdots a_r}{}_{b_1\cdots b_s} - \sum_{q=1}^{s} \bar R^{t}{}_{b_q ab}\,\Xi^{a_1\cdots a_r}{}_{b_1\cdots b_{q-1}tb_{q+1}\cdots b_s},$$

where we must replace the tensor $\Xi^{a_1\cdots a_r}{}_{b_1\cdots b_s}$ by the system variables and apply (8) to work out the covariant derivatives. Each one of the equations derived in this fashion only involves system variables and it is called *first integrability condition*. These integrability conditions can be further covariantly differentiated yielding integrability conditions of higher degree. The constraint equations (11) also give rise to integrability conditions when differentiated in the obvious way.

3.1. Eq. (8f)

This is the simplest integrability condition being its expression

$$\bar\nabla_c\bar\nabla_b\xi^a - \bar\nabla_b\bar\nabla_c\xi^a = \bar R^a{}_{rcb}\xi^r = \bar\nabla_c\Psi_b{}^a - \bar\nabla_b\Psi_c{}^a,$$

which is an identity as is easily checked by replacing the covariant derivatives of $\Psi_a{}^b$.

3.2. Eq. (8g)

The integrability conditions of (8g) are given by

$$\begin{aligned}\bar\nabla_a\bar\nabla_b\Psi_c{}^d - \bar\nabla_b\bar\nabla_a\Psi_c{}^d = {}& -\Psi_a{}^r\bar R^d{}_{crb} + \Psi_b{}^r\bar R^d{}_{cra} - \xi^r\bar\nabla_r\bar R^d{}_{cab}\\ &+ \tfrac{1}{2}\nabla_a((\bar\phi_b P^d{}_c + \bar\phi_c P^d{}_b - \bar\phi^d P_{cb} + \bar\chi_b\Pi^d{}_c + \bar\chi_c\Pi^d{}_b - \bar\chi^d\Pi_{cb}))\\ &- \tfrac{1}{2}\nabla_b((\bar\phi_a P^d{}_c + \bar\phi_c P^d{}_a - \bar\phi^d P_{ca} + \bar\chi_a\Pi^d{}_c + \bar\chi_c\Pi^d{}_a - \bar\chi^d\Pi_{ca})).\end{aligned}$$

We apply now the Ricci identity to the left-hand side of this expression and gather all the terms containing contractions with the tensor $\Psi_a{}^b$ in a single term by means of the identity

$$\pounds_{\vec{\xi}}\, \bar R^d{}_{cab} = \xi^r\bar\nabla_r\bar R^d{}_{cab} - \Psi_r{}^d\bar R^r{}_{cab} + \Psi_c{}^r\bar R^d{}_{rab} + \Psi_a{}^r\bar R^d{}_{crb} + \Psi_b{}^r\bar R^d{}_{car},$$

from which we obtain

$$\begin{aligned}\pounds_{\vec{\xi}}\, \bar R^d{}_{cab} = {}& \bar\nabla_{[a}\bar\phi_{b]}P^d{}_c + P^d{}_{[b}\bar\nabla_{a]}\bar\phi_c - P_{c[b}\bar\nabla_{a]}\bar\phi^d + \bar\nabla_{[a}\bar\chi_{b]}\Pi^d{}_c\\ &+ \Pi^d{}_{[b}\bar\nabla_{a]}\bar\chi_c - \Pi_{c[b}\bar\nabla_{a]}\bar\chi^d + \bar\phi_{[b}\bar\nabla_{a]}P^d{}_c + \bar\phi_c\bar\nabla_{[a}P^d{}_{b]}\\ &- \bar\phi^d\bar\nabla_{[a}P_{b]c} + \bar\chi_{[b}\bar\nabla_{a]}\Pi^d{}_c + \bar\chi_c\bar\nabla_{[a}\Pi^d{}_{b]} - \bar\chi^d\bar\nabla_{[a}\Pi_{b]c}.\end{aligned} \tag{12}$$

The direct substitution of $\bar{\nabla}_a\bar{\chi}_b$, $\bar{\nabla}_a\bar{\phi}_b$ by the expressions given by (8d) and (8e), respectively, yields after lengthy algebra

$$\frac{1}{2}\,\pounds_{\vec{\xi}}\,T^d{}_{cab} = \frac{\bar{\phi}_r}{2-p}(P^d{}_{[b}\Lambda^r_{a]c} + P^d{}_q\Upsilon^{rq}{}_{[b}P_{a]c}) + \frac{\bar{\chi}_r}{2-(n-p)} \times(\Pi^d{}_{[b}\bar{\Lambda}^r{}_{a]c} + \Pi^d{}_q\bar{\Upsilon}^{rq}{}_{[b}P_{a]c}) + \bar{\phi}_{[b}\bar{\nabla}_{a]}P^d{}_c + \bar{\phi}_c\bar{\nabla}_{[a}P^d{}_{b]} + \bar{\phi}^d\bar{\nabla}_{[b}P_{a]c} + \bar{\chi}^d\bar{\nabla}_{[b}\Pi_{a]c} + \bar{\chi}_{[b}\bar{\nabla}_{a]}\Pi^d{}_c + \bar{\chi}_c\bar{\nabla}_{[a}\Pi^d{}_{b]}, \tag{13}$$

where by convenience we introduce the tensors

$$\Lambda^d{}_{bc} \equiv 2P^{dr}\bar{\nabla}_rP_{bc},\qquad \bar{\Lambda}^d{}_{bc} = 2\Pi^{dr}\bar{\nabla}_r\Pi_{bc},\qquad \Upsilon^{sc}{}_b \equiv 2P^{sr}P^{cq}\bar{\nabla}_rP_{qb} + (2-p)\bar{\nabla}_bP^{sc}, \tag{14}$$

$$\bar{\Upsilon}^{sc}{}_b \equiv 2\Pi^{sr}\Pi^{cq}\bar{\nabla}_r\Pi_{qb} + (2-n+p)\bar{\nabla}_b\Pi^{sc}, \tag{15}$$

and

$$T^d{}_{cab} \equiv 2\bar{R}^d{}_{cab} - \frac{2}{2-p}(P^d{}_cL^0_{[ab]} + P^d{}_{[b}L^0_{a]c} + P_{c[a}L^0_{b]q}P^{qd}) - \frac{2}{2-n+p}(\Pi^d{}_cL^1_{[ab]} + \Pi^d{}_{[b}L^1_{a]c} + \Pi_{c[a}L^1_{b]q}\Pi^{qd}). \tag{16}$$

This last tensor will play an important role in the local characterization of bi-conformally flat pseudo-Riemannian manifolds as will be seen later.

An interesting invariance property of some of the above tensors needed in future calculations is

$$\pounds_{\vec{\xi}}\,\Lambda^d{}_{bc} = 0,\qquad \pounds_{\vec{\xi}}\,\bar{\Lambda}^d{}_{bc} = 0 \tag{17}$$

which are easily obtained from (26).

3.3. Eq. (8b) and (8c)

Again the calculations are tedious but straightforward. The covariant derivatives of $\bar{\phi}_a$ and $\bar{\chi}_a$ are calculated through (8d) and (8e) and to commute the Lie derivative and the covariant derivative when differentiating both equations we use the identity (see [19,15])

$$\bar{\nabla}_c\,\pounds_{\vec{\xi}}\,T^{a_1\cdots a_s}{}_{b_1\cdots b_q} - \pounds_{\vec{\xi}}\,\bar{\nabla}_cT^{a_1\cdots a_s}{}_{b_1\cdots b_q} = -\sum_{j=1}^{s}(\pounds_{\vec{\xi}}\,\bar{\gamma}^{a_j}_{cr})T^{\cdots a_{j-1}ra_{j+1}\cdots}{}_{b_1\cdots b_q} + \sum_{j=1}^{q}(\pounds_{\vec{\xi}}\,\bar{\gamma}^r_{cb_j})T^{a_1\cdots a_s}{}_{\cdots b_{j-1}rb_{j+1}\cdots}, \tag{18}$$

where, as calculated in paper I, the Lie derivative of the bi-conformal connection is

$$\pounds_{\vec{\xi}}\,\bar{\gamma}^a{}_{bc} = \frac{1}{2}(\bar{\phi}_bP^a{}_c + \bar{\phi}_cP^a{}_b - \bar{\phi}^aP_{cb} + \bar{\chi}_b\Pi^a{}_c + \bar{\chi}_c\Pi^a{}_b - \bar{\chi}^a\Pi_{cb}).$$

Recall that the Lie derivative of a connection is always a tensor even though the connection itself is not (see e.g. [19]). Putting all this together we get

$$
\begin{aligned}
E_d\, \pounds_{\vec{\xi}}\, (\Pi^d{}_r T^r{}_{acb}) = {}& \bar{\chi}_a \bar{\nabla}_{[c} E_{b]} + \bar{\chi}_{[b} \bar{\nabla}_{c]} E_a - \bar{\chi}^r \bar{\nabla}_{[c}(\Pi_{b]a} E_r) \\
& + (P^r{}_a \bar{\phi}_{[c} + \bar{\phi}_a P^r{}_{[c} - \bar{\phi}^r P_{a[c} + \Pi^r{}_a \bar{\chi}_{[c} + \bar{\chi}_a \Pi^r{}_{[c} - \bar{\chi}^r \Pi_{a[c}) \bar{\nabla}_{b]} E_r \\
& + \frac{1}{2-n+p} \bar{\chi}_q E_r \bar{\Upsilon}^{qr}{}_{[b} \Pi_{c]a},
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
W_d\, \pounds_{\vec{\xi}}\, (P^d{}_r T^r{}_{acb}) = {}& \bar{\phi}_a \bar{\nabla}_{[c} W_{b]} + \bar{\phi}_{[b} \bar{\nabla}_{c]} W_a - \bar{\phi}^r \bar{\nabla}_{[c}(P_{b]a} W_r) \\
& + (\Pi^r{}_a \bar{\chi}_{[c} + \bar{\chi}_a \Pi^r{}_{[c} - \bar{\chi}^r \Pi_{a[c} + P^r{}_a \bar{\phi}_{[c} + \bar{\phi}_a P^r{}_{[c} - \bar{\phi}^r P_{a[c}) \bar{\nabla}_{b]} W_r \\
& + \frac{1}{2-p} \bar{\phi}_q W_r \Upsilon^{qr}{}_{[b} P_{c]a}.
\end{aligned} \tag{20}
$$

3.4. Eq. (8a)

The integrability conditions of this equation are easier to handle

$$
\begin{aligned}
0 &= \bar{\nabla}_a \bar{\nabla}_b \phi - \bar{\nabla}_b \bar{\nabla}_a \phi = \bar{\nabla}_a \bar{\phi}_b - \bar{\nabla}_b \bar{\phi}_a + \bar{\nabla}_a \phi^*_b - \bar{\nabla}_b \phi^*_a, \\
0 &= \bar{\nabla}_a \bar{\nabla}_b \chi - \bar{\nabla}_b \bar{\nabla}_a \chi = \bar{\nabla}_a \bar{\chi}_b - \bar{\nabla}_b \bar{\chi}_a + \bar{\nabla}_a \chi^*_b - \bar{\nabla}_b \chi^*_a,
\end{aligned}
$$

from which we readily obtain by means of (8b)–(8e)

$$
\pounds_{\vec{\xi}} \left(\frac{1}{2-p} L^0_{[ab]} - \frac{1}{p} \bar{\nabla}_{[a} E_{b]} \right) = 0, \tag{21}
$$

$$
\pounds_{\vec{\xi}} \left(\frac{1}{2-n+p} L^1_{[ab]} - \frac{1}{n-p} \bar{\nabla}_{[a} W_{b]} \right) = 0. \tag{22}
$$

3.5. Eqs. (8d) and (8e)

The integrability conditions yield (Eq. (17) is used along the way)

$$
\begin{aligned}
\frac{2-p}{2} \bar{\phi}_r T^r{}_{ceb} = {}& 2\, \pounds_{\vec{\xi}} \left(\bar{\nabla}_{[e} L^0_{b]c} + \frac{1}{2-p} \Lambda^d{}_{c[b} L^0_{e]d} \right) + \bar{\chi}_{[e} L^0_{b]q} \Pi^q{}_c + \bar{\chi}_c \Pi^q{}_{[e} L^0_{b]q} \\
& - \bar{\chi}^q \Pi_{c[e} L^0_{b]q} + \frac{2}{2-p} \bar{\phi}_r (\Lambda^d_{c[b} \Lambda^r_{e]d} + (2-p) \bar{\nabla}_{[e} \Lambda^r{}_{b]c}),
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
\frac{2-n+p}{2} \bar{\chi}_r T^r{}_{ceb} = {}& 2\, \pounds_{\vec{\xi}} \left(\bar{\nabla}_{[e} L^1_{b]c} + \frac{1}{2-n-p} \bar{\Lambda}^d{}_{c[b} L^1_{e]d} \right) + \bar{\chi}_{[e} L^1_{b]q} P^q{}_c + \bar{\phi}_c P^q{}_{[e} L^1_{b]q} \\
& - \bar{\phi}^q P_{c[e} L^1_{b]q} + \frac{2}{2-n+p} \bar{\chi}_r (\bar{\Lambda}^d_{c[b} \bar{\Lambda}^r_{e]d} + (2-n+p) \bar{\nabla}_{[e} \bar{\Lambda}^r{}_{b]c}).
\end{aligned} \tag{24}
$$

3.6. Constraint equations

Finally only the first integrability conditions coming up from the constraints (11) are left. These integrability conditions result from their covariant derivative (we only need to take care of (11a) because the differentiation of (11b) results in (8b) and (8c) which are part of the normal form). Differentiation of these equations with respect to $\bar{\nabla}$ yields after some algebra (apply identity (18))

$$£_{\vec{\xi}}\,\bar{\nabla}_c P_{ab} = \phi\bar{\nabla}_c P_{ab} + \phi^*_c P_{ab}, \qquad £_{\vec{\xi}}\,\bar{\nabla}_c \Pi_{ab} = \chi\bar{\nabla}_c \Pi_{ab} + \chi^*_c \Pi_{ab}. \tag{25}$$

For completeness we provide also these equations with the *ab* indexes raised

$$£_{\vec{\xi}}\,\bar{\nabla}_c P^{ab} = -\phi^*_c P^{ab} - \phi\bar{\nabla}_c P^{ab}, \qquad £_{\vec{\xi}}\,\bar{\nabla}_c \Pi^{ab} = -\chi^*_c \Pi^{ab} - \chi\bar{\nabla}_c \Pi^{ab}, \tag{26}$$

and the invariance laws

$$£_{\vec{\xi}}\,\bar{\nabla}_c P^a{}_b = 0, \qquad £_{\vec{\xi}}\,\bar{\nabla}_c \Pi^a{}_b = 0. \tag{27}$$

These equations close the whole suite of first integrability conditions. In the next section we will obtain geometric information from these conditions.

4. Complete integrability

Our next task is to find out when the first integrability conditions presented in the previous section become a set of identities for every choice of the independent variables of the normal form (8). This happens if certain geometric conditions (complete integrability conditions) are met. Under such conditions there exists in the neighbourhood of each point a finite-dimensional Lie algebra of bi-conformal vector fields attaining the greatest dimension *N*. Thus if we are able to find pseudo-Riemannian manifolds in which these conditions are satisfied we will have proven that the bound *N* is reached at least locally (see [4]). The first problem which we come across to is that not all the variables appearing in (8) are independent as there are constraints which must be taken into account. However, some of the variables of (8) are not involved in the constraint equations (11) and this fact will allow us to find necessary and sufficient geometric conditions for the whole set of first integrability conditions to become identities. The variables which are not constrained by (11) are $\bar{\phi}_a$ and $\bar{\chi}_a$ so we will separate out in each of the integrability conditions obtained above all the contributions involving these variables. If we demand then that $\bar{\phi}_a$ and $\bar{\chi}_a$ be arbitrary functions in the neighbourhood of a point we will obtain a number of local geometric conditions which will lead us to the complete integrability conditions.

We will perform next this procedure step by step (actually we will not need to analyse all the conditions as some of them will turn into identities if the geometric conditions entailed by others are imposed). The full calculation is rather cumbersome and only its main excerpts will be shown so the reader interested in the complete integrability conditions should jump directly to Theorem 9.

4.1. Eq. (13)

This equation can be rewritten as

$$£_{\vec{\xi}}\,T^d{}_{cab} = \bar{\phi}_r M^{rd}{}_{cab} + \bar{\chi}_r N^{rd}{}_{cab}, \tag{28}$$

where

$$M^{rd}{}_{cab} \equiv \frac{2}{2-p} P^r{}_s(\Lambda^s{}_{c[a} P^d{}_{b]} + \Upsilon^{sd}{}_{[b} P_{a]c}) + 2P^r{}_{[b}\bar{\nabla}_{a]} P^d{}_c$$
$$+ 2P^r{}_c \bar{\nabla}_{[a} P^d{}_{b]} + 2P^{dr}\bar{\nabla}_{[b} P_{a]c}, \tag{29}$$

$$N^{rd}{}_{cab} \equiv \frac{2}{2-n+p}\Pi^r{}_s(\bar{\Lambda}^s{}_{c[a}\Pi^d{}_{b]} + \bar{\Upsilon}^{sd}{}_{[b}\Pi_{a]c}) + 2\Pi^r{}_{[b}\bar{\nabla}_{a]}\Pi^d{}_c$$
$$+ 2\Pi^r{}_c\bar{\nabla}_{[a}\Pi^d{}_{b]} + 2\Pi^{dr}\bar{\nabla}_{[b}\Pi_{a]c}. \tag{30}$$

Under the assumption of complete integrability (28) must be true for every $\bar{\phi}_a$, $\bar{\chi}_a$ so we have

$$P^r{}_sM^{sd}{}_{cab} = M^{rd}{}_{cab} = 0, \qquad \Pi^r{}_sN^{sd}{}_{cab} = N^{rd}{}_{cab} = 0. \tag{31}$$

Let us study these geometric conditions (it is enough to concentrate on the first condition because the other is dual and it is obtained by the usual replacements). Contracting the indexes d and b in this equation we get

$$\frac{2}{2-p}[pP^{rq}\bar{\nabla}_qP_{ac} - P^{rq}(P_a{}^s\bar{\nabla}_qP_{sc} + P_c{}^s\bar{\nabla}_qP_{sa})] + P^{sr}\bar{\nabla}_sP_{ac} = 0. \tag{32}$$

Multiplying this by $P_z{}^a$ gives the condition

$$P_z{}^sP^{rq}\bar{\nabla}_qP_{sc} = 0, \tag{33}$$

which put back in (32) yields

$$P^{rs}\bar{\nabla}_sP_{ac} = 0 \Rightarrow \Lambda^r{}_{ac} = 0, \qquad \Upsilon^{sc}{}_b = (2-p)\bar{\nabla}_bP^{sc}. \tag{34}$$

Now using this result we contract r and b in the first of (31)

$$(p-1)\bar{\nabla}_aP^d{}_c - P_a{}^s\bar{\nabla}_sP^d{}_c - P_c{}^s\bar{\nabla}_sP^d{}_a + P_c{}^s\bar{\nabla}_aP^d{}_s = 0, \tag{35}$$

from which we get

$$P_a{}^qP_c{}^s\bar{\nabla}_qP^d{}_s = 0. \tag{36}$$

Combining this with (35) we readily obtain

$$\bar{\nabla}_aP^d{}_c = 0. \tag{37}$$

Finally multiplying by P^{ac} in the first of (31) gives

$$P^r{}_s(-P^a{}_b\bar{\nabla}_aP^{sd} + p\bar{\nabla}_bP^{sd}) - P^{dr}E_b = 0. \tag{38}$$

Multiplying this last equation by $P_z{}^b$ we obtain

$$P^a{}_zP^r{}_s\bar{\nabla}_aP^{sd} = 0,$$

which combined with (38) implies

$$\bar{\nabla}_bP^{rd} = \frac{1}{p}E_bP^{rd}. \tag{39}$$

This last condition can be written with the indexes of the projector lowered if we use (37)

$$\bar{\nabla}_bP_{cd} = -\frac{1}{p}E_bP_{cd}. \tag{40}$$

The dual conditions for the projectors Π^{ab}, Π_{ab} coming from the vanishing of (30) are

$$\bar{\nabla}_c\Pi^{ab} = \frac{1}{n-p}W_c\Pi^{ab}, \qquad \bar{\nabla}_c\Pi^a{}_b = 0, \qquad \bar{\nabla}_c\Pi_{ab} = -\frac{1}{p}W_c\Pi_{ab}. \tag{41}$$

Conversely, if Eqs. (37), (39), (40) and (41) are assumed a simple calculation tells us that the tensors defined by (29) and (30) vanish. In fact all the above geometric conditions can be combined in a single simpler expression.

Proposition 2. *The following assertion is true*

$$M_{abc} = \frac{1}{p} E_a P_{bc} - \frac{1}{n-p} W_a \Pi_{bc} \Longleftrightarrow \tag{42}$$

$$\bar{\nabla}_a P_{bc} = -\frac{1}{p} E_a P_{bc}, \qquad \bar{\nabla}_a \Pi_{bc} = -\frac{1}{n-p} W_a \Pi_{bc}. \tag{43}$$

Proof. To prove this result we need the following formulae relating $\bar{\nabla}_c P_{ab}$, $\bar{\nabla}_c P^a{}_b$ and $\nabla_c P_{ab}$, $\nabla_c P^a{}_b$.

$$\bar{\nabla}_a P_{bc} = \nabla_a P_{bc} - \frac{1}{p} E_a P_{bc} - \frac{1}{2p}(E_b P_{ac} + E_c P_{ab}) - \frac{1}{2}(P_{cp} M^p{}_{ab} + P_{bp} M^p{}_{ac}), \tag{44}$$

$$2\bar{\nabla}_a P^b{}_c = 2\nabla_a P^b{}_c + P^{bq} P^r{}_c M_{qra} - \Pi^{bq} P^r{}_c M_{qra} - P^b{}_q M^q{}_{ac} + \frac{1}{n-p} W_c \Pi^b{}_a - \frac{1}{p} E_c P^b{}_a, \tag{45}$$

$$\bar{\nabla}_a P^{bc} = \nabla_a P^{bc} + \frac{1}{p} E_a P^{bc} + \frac{1}{2(n-p)}(W^c \Pi^b{}_a + W^b \Pi^c{}_a) - \frac{1}{2}(M^b{}_{ar} P^{rc} + M^c{}_{ar} P^{rb}). \tag{46}$$

There is a dual set of identities obtained through the replacements $P_{ab} \leftrightarrow \Pi_{ab}$, $E_a \leftrightarrow W_a$ and $p \leftrightarrow n-p$ (see paper I for a proof). Now, if conditions (43) are true then expanding $\bar{\nabla}_a P_{bc}$ by means of (44) we get

$$\nabla_b P_{ac} = \frac{1}{2p}(E_a P_{bc} + E_c P_{ab}) + \frac{1}{2}(P_{cp} M^p{}_{ba} + P_{ap} M^p{}_{bc}), \tag{47}$$

$$\nabla_b \Pi_{ac} = \frac{1}{2(n-p)}(W_a \Pi_{bc} + W_c \Pi_{ab}) - \frac{1}{2}(\Pi_{cp} M^p{}_{ba} + \Pi_{ap} M^p{}_{bc}). \tag{48}$$

Substituting these expressions of the covariant derivatives of P_{ab} and Π_{ab} in the definition of M_{abc} (Eq. (7)) yields

$$P_{cp} M^p{}_{ab} = -\frac{1}{n-p} W_c \Pi_{ab}, \qquad \Pi_{cp} M^p{}_{ab} = \frac{1}{p} E_c P_{ab}, \tag{49}$$

whose addition leads to (42). Conversely, suppose that (42) holds. We may write this equation in the equivalent form

$$\nabla_b P_{ac} = \frac{1}{2p}(E_a P_{bc} + E_c P_{ab}) - \frac{1}{2(n-p)}(W_a \Pi_{bc} + W_c \Pi_{ba}), \tag{50}$$

where the relation $M_{abc} = \nabla_b P_{ac} - \nabla_c P_{ab} - \nabla_a P_{bc}$ has been again used. Then inserting (50) and (42) into (44) gives us the condition on $\bar{\nabla}_a P_{bc}$ at once. The calculation for $\bar{\nabla}_a \Pi_{bc}$ is similar using identity (44) written in terms of Π_{ab}. □

Proposition 3. *If* (42) *is true* $\bar{\nabla}_c P^a{}_b = 0$.

Proof. To prove this we use identity (45) and replace M_{abc} by (42) yielding

$$\bar{\nabla}_a P^b{}_c = \nabla_a P^b{}_c - \frac{1}{2p}(E^b P_{ac} + E_c P^b{}_a) + \frac{1}{2(n-p)}(W_c \Pi^b{}_a + W^b \Pi_{ac}), \tag{51}$$

which vanishes by (50). □

Remark 4. Note that in view of the above result condition (42) entails

$$\bar{\nabla}_c P^{ab} = \frac{1}{p} E_c P^{ab}, \qquad \bar{\nabla}_c \Pi^{ab} = \frac{1}{n-p} W_c \Pi^{ab}. \tag{52}$$

Therefore all the conditions coming from (29) and (30) are summarized by (42). This last equation can be written in the equivalent form $T_{abc} = 0$, where

$$T_{abc} \equiv M_{abc} - \frac{1}{p} E_a P_{bc} + \frac{1}{n-p} W_a \Pi_{bc}.$$

As explained in paper I the tensor T_{abc} plays a key role in the geometric characterization of *conformally separable pseudo-Riemannian manifolds*. There we proved that a pseudo-Riemannian manifold is locally conformally separable with the tensors P_{ab} and Π_{ab} as the leaf metrics (see Definition 10) if and only if $T_{abc} = 0$ which means that the complete integrability conditions obtained so far bear a clear geometrical meaning.

Some of the first integrability conditions achieve a great simplification if T_{abc} vanishes. For instance (25)–(27) become zero identically under this condition as is obvious from Propositions 2 and 3 so we do not need to care about these integrability conditions any more. Eq. (28) acquires the invariance law

$$\pounds_{\vec{\xi}}\, T^d{}_{cab} = 0. \tag{53}$$

Other simplifications will be shown in the forthcoming analysis.

4.2. Eqs. (23) and (24)

If $T_{abc} = 0$ then $\Lambda^a{}_{bc} = 0$, $\bar{\Lambda}^a{}_{bc} = 0$ so these equations take the form

$$-\bar{\phi}_q P^q{}_r T^r{}_{ceb} = 2\, \pounds_{\vec{\xi}}\, \bar{\nabla}_{[e} L^0_{b]c} + \bar{\chi}_r E^r{}_{ceb}, \tag{54}$$

$$-\bar{\chi}_q \Pi^q{}_r T^r{}_{ceb} = 2\, \pounds_{\vec{\xi}}\, \bar{\nabla}_{[e} L^1_{b]c} + \bar{\phi}_r F^r{}_{ceb}, \tag{55}$$

where

$$E^r{}_{ceb} = \frac{2}{2-p}(\Pi^q{}_c \Pi^r{}_{[e} L^0{}_{b]q} + \Pi^r{}_c \Pi^q{}_{[e} L^0{}_{b]q} - \Pi^{rq} \Pi_{c[e} L^0_{b]q}), \tag{56}$$

$$F^r{}_{ceb} = \frac{2}{2-n+p}(P^q{}_c P^r{}_{[e} L^1{}_{b]q} + P^r{}_c P^q{}_{[e} L^1_{b]q} - P^{rq} P_{c[e} L^1_{b]q}). \tag{57}$$

If (54) and (55) are to be true for every value of $\bar{\phi}_q$ and $\bar{\chi}_q$ we find the conditions of complete integrability

$$P^q{}_r T^r{}_{ceb} = 0, \qquad \Pi^q{}_r T^r{}_{ceb} = 0 \Longrightarrow T^q{}_{ceb} = 0,$$

so Eq. (53) is trivially fulfilled. To proceed further with the calculations we need a lemma.

Lemma 5. *If* $\bar{\nabla}_a P^b{}_c = 0$ *then*

$$L^0_{bq}\Pi^q{}_c = 0, \qquad L^1_{bq}P^q{}_c = 0.$$

Proof. These properties are proven through the Ricci identity applied to the tensors $P^b{}_c$, $\Pi^b{}_c$ which take a remarkably simple form under our conditions (we only perform the calculations for the tensor $P^a{}_b$)

$$0 = \bar{\nabla}_e\bar{\nabla}_a P^b{}_c - \bar{\nabla}_a\bar{\nabla}_e P^b{}_c = \bar{R}^b{}_{qea}P^q{}_c - \bar{R}^q{}_{cea}P^b{}_q, \tag{58}$$

whence

$$L^0_{bq}\Pi^q{}_c = 2P^d{}_r\bar{R}^r{}_{qdb}\Pi^q{}_c + \frac{2}{p}(\Pi^d{}_b P^r{}_q \bar{R}^q{}_{rds}\Pi^s{}_c).$$

The first term of this expression is zero according to (58) and the second one can be transformed by means of the first Bianchi identity into

$$\begin{aligned}\Pi^d{}_b\Pi^s{}_c P^r{}_q\bar{R}^q{}_{rds} &= -\Pi^d{}_b\Pi^s{}_c(P^r{}_q\bar{R}^q{}_{dsr} + P^r{}_q\bar{R}^q{}_{srd})\\ &= -\Pi^d{}_b\Pi^s{}_c(P^q{}_d\bar{R}^r{}_{qsr} + P^q{}_s\bar{R}^r{}_{qrd}),\end{aligned}$$

which also vanishes. □

Therefore conditions (54) and (55) are further simplified to

$$\pounds_{\vec{\xi}}\,\bar{\nabla}_{[e}L^0_{b]c} = 0, \qquad \pounds_{\vec{\xi}}\,\bar{\nabla}_{[e}L^1_{b]c} = 0. \tag{59}$$

It is our next aim to show that indeed these two equations are identities if $T^d{}_{cab} = 0$.

Lemma 6. *If* $p \neq 3$, $n - p \neq 3$ *and* $T_{abc} = 0$ *then*

$$T^d{}_{cab} = 0 \Longrightarrow \bar{\nabla}_{[e}L^0_{b]c} = 0, \qquad \bar{\nabla}_{[e}L^1_{b]c} = 0.$$

Proof. To prove this we start from the identity

$$\begin{aligned}&(2-p)[\bar{\nabla}_e(P^e{}_qT^q{}_{cab}) + \bar{\nabla}_b(P^d{}_qT^q{}_{cda}) - \bar{\nabla}_a(P^d{}_qT^q{}_{cdb})]\\ &\quad = 2P^e{}_c\bar{\nabla}_eL^0_{[ba]} - 2P^d{}_c\bar{\nabla}_{[b|}L^0_{d|a]} + 2p\bar{\nabla}_{[b}L^0_{a]c} + 2P^d{}_{[b}\bar{\nabla}_{a]}L^0_{dc} - 2P^e{}_{[b|}\bar{\nabla}_eL^0_{|a]c}\\ &\qquad + 2P^{qe}\bar{\nabla}_eL^0_{[a|q}P_{|b]c} + 2P^{qd}P_{c[a}\bar{\nabla}_{b]}L^0_{dq},\end{aligned} \tag{60}$$

which is obtained from Eq. (16) and the second Bianchi identity for the tensor $\bar{R}^a{}_{bcd}$ if the condition $T_{abc} = 0$ holds. By assumption the left-hand side of this identity vanishes so we only need to study the right-hand side equated to zero. Multiplying such equation by P^{ca} we get

$$2(1-p)P^{ae}\bar{\nabla}_eL^0_{ba} - 2(1-p)P^{da}\bar{\nabla}_bL^0_{da} + 2P^d{}_bP^{ac}\bar{\nabla}_aL^0_{dc} - 2P^e{}_bP^{ac}\bar{\nabla}_eL^0_{ac} = 0, \tag{61}$$

and a further multiplication by $P_z{}^b$ yields

$$P_z{}^rP^{qe}\bar{\nabla}_eL^0_{rq} = P_z{}^rP^{qd}\bar{\nabla}_rP_{qd}.$$

This last property implies that the last two terms of (61) are zero from which we conclude that

$$P^{qe}\bar{\nabla}_eL^0_{rq} = P^{qd}\bar{\nabla}_rP_{qd},$$

and hence (60) becomes

$$2P^e{}_c\bar{\nabla}_eL^0_{[ba]} - 2P^d{}_c\bar{\nabla}_{[b|}L^0_{d|a]} + 2p\bar{\nabla}_{[b}L^0_{a]c} + 2P^d{}_{[b}\bar{\nabla}_{a]}L^0_{dc} = 0. \tag{62}$$

If we multiply this last equation by $P_z{}^a P_t{}^b$ we obtain

$$S_{ctz} - S_{tcz} + S_{zct} - S_{czt} + (p-2)(S_{tzc} - S_{ztc}) = 0, \tag{63}$$

where

$$S_{ztc} \equiv P_z{}^r P_t{}^s \bar{\nabla}_r L^0_{sc}.$$

Permuting indexes in (63) we get the equations

$$\begin{aligned} &S_{ctz} - S_{tcz} + S_{zct} - S_{czt} + (p-2)(S_{tzc} - S_{ztc}) = 0,\\ &S_{ztc} - S_{tzc} + S_{czt} - S_{zct} + (p-2)(S_{tcz} - S_{ctz}) = 0,\\ &S_{ctz} - S_{tcz} + S_{tzc} - S_{ztc} + (p-2)(S_{zct} - S_{czt}) = 0 \end{aligned} \tag{64}$$

Setting the variables $x = S_{ctz} - S_{tcz}$, $y = S_{zct} - S_{czt}$, $w = S_{tzc} - S_{ztc}$ we deduce that previous equations form a homogeneous system in these variables whose matrix is

$$\begin{pmatrix} 1 & 1 & p-2 \\ 2-p & -1 & -1 \\ 1 & p-2 & 1 \end{pmatrix} \Rightarrow \begin{vmatrix} 1 & 1 & p-2 \\ 2-p & -1 & -1 \\ 1 & p-2 & 1 \end{vmatrix} = -p(p-3)^2.$$

So unless $p = 3$ ($p = 0$ makes no sense in the current context) we conclude that $x = y = w = 0$ and hence

$$P_{[a}{}^r P_{b]}{}^s \bar{\nabla}_r L^0_{sc} = 0.$$

Application of this in the expression resulting of multiplying (62) by $P^r{}_b$ leads to

$$P^d{}_c \bar{\nabla}_a L^0_{db} - P^e{}_c \bar{\nabla}_e L^0_{ab} + (p-1)(P^r{}_b \bar{\nabla}_r L^0_{ac} - P^r{}_b \bar{\nabla}_a L^0_{rc}) = 0.$$

By setting $Q_{acb} \equiv P^d{}_c \bar{\nabla}_a L^0_{db} - P^e{}_c \bar{\nabla}_e L^0_{ab}$ we can rewrite this as

$$Q_{acb} - (p-1)Q_{abc} = 0, \qquad \Rightarrow Q_{abc} - (p-1)Q_{acb} = 0,$$

which entails $Q_{abc} = 0$ (recall that $p \neq 1$ by definition of L^0_{ab}). This last property applied to (62) yields

$$\bar{\nabla}_{[b} L^0_{a]c} = 0,$$

as desired. The result for L^1_{ab} is proven in a similar way. □

4.3. Eqs. (21) and (22)

The analysis of these conditions is performed by means of the following result.

Proposition 7. *If* $T_{abc} = 0$ *then*

$$\frac{1}{2-p}(L^0_{ab} - L^0_{ba}) - \frac{1}{p}(\bar{\nabla}_a E_b - \bar{\nabla}_b E_a) = 0,$$

$$\frac{1}{2-n+p}(L^1_{ab} - L^1_{ba}) - \frac{1}{n-p}(\bar{\nabla}_a W_b - \bar{\nabla}_b W_a) = 0.$$

Proof. We only carry on the proof for the first identity as the calculations are similar for the second one. We start from the identity

$$E_a = -P^{bc}\bar{\nabla}_a P_{bc},$$

which is easily obtained from (44). Using this we may write

$$\bar{\nabla}_a E_b - \bar{\nabla}_b E_a = -P^{qr}(\bar{\nabla}_a\bar{\nabla}_b P_{qr} - \bar{\nabla}_b\bar{\nabla}_a P_{qr}) - \bar{\nabla}_a P^{qr}\bar{\nabla}_b P_{qr} + \bar{\nabla}_b P^{qr}\bar{\nabla}_a P_{qr}. \quad (65)$$

The expression in brackets can be transformed by the Ricci identity into $2P^q{}_r\bar{R}^r{}_{qab}$. If we impose now the condition $T_{abc}=0$, then combination of Proposition 2 and Remark 4 entails

$$\bar{\nabla}_a P^{qr}\bar{\nabla}_b P_{qr} = -\frac{1}{p}E_a E_b = \bar{\nabla}_b P^{qr}\bar{\nabla}_a P_{qr}.$$

Therefore after these manipulations equation (65) yields

$$\bar{\nabla}_a E_b - \bar{\nabla}_b E_a = 2P^q{}_r\bar{R}^r{}_{qab}. \quad (66)$$

On the other hand from (9) and applying the first Bianchi identity it is easy to obtain

$$L^0_{ab} - L^0_{ba} = \frac{2(2-p)}{p}P^r{}_q\bar{R}^q{}_{rab}. \quad (67)$$

Combination of (66) and (67) leads to the desired result. □

4.4. Eqs. (19) and (20)

Proposition 8. *If $T^d{}_{cab}=0$ and $T_{abc}=0$ then (19) and (20) are identities.*

Proof. The left-hand side of both equations vanishes trivially if $T^d{}_{cab}=0$ so we just need to show that the right-hand side vanishes as well. The characterization of the condition $T_{abc}=0$ in terms of the covariant derivatives of the projectors (Eqs. (43) and (52)) entails

$$\Upsilon^{sc}{}_b = \frac{2-p}{p}E_b P^{sc}, \qquad \bar{\Upsilon}^{sc}{}_b = \frac{2-n+p}{n-p}W_b\Pi^{sc},$$

which means that the terms of (19) and (20) containing these tensors are zero. The property $\bar{\nabla}_c P^a{}_b = \bar{\nabla}_c \Pi^a{}_b = 0$ can be used now to get rid of some terms and simplify others on these couple of equations getting

$$0 = \bar{\chi}_a\bar{\nabla}_{[c}E_{b]} + \bar{\chi}_{[b}\bar{\nabla}_{c]}E_a + \bar{\chi}_a\bar{\nabla}_{[b}E_{c]} + \bar{\chi}_{[c}\bar{\nabla}_{b]}E_a,$$
$$0 = \bar{\phi}_a\bar{\nabla}_{[c}W_{b]} + \bar{\phi}_{[b}\bar{\nabla}_{c]}W_a + \bar{\phi}_a\bar{\nabla}_{[b}W_{c]} + \bar{\phi}_{[c}\bar{\nabla}_{b]}W_a,$$

which are obviously identities. □

4.5. Constraints

If $T_{abc}=0$ then using (43) and Proposition 3 Eq. (25) becomes

$$-\frac{1}{p}\pounds_{\vec{\xi}}(E_c P_{ab}) = -\frac{\phi}{p}E_c P_{ab} + \phi^*_c P_{ab}, \quad -\frac{1}{n-p}\pounds_{\vec{\xi}}(W_c\Pi_{ab}) = \frac{-\chi}{n-p}W_c\Pi_{ab} + \chi^*_c\Pi_{ab},$$

which is easily seen to be an identity if we apply (11b).

All our calculations are thus summarized in the next result, which is one of the most important of this paper.

Theorem 9 (Complete integrability conditions). *The first integrability conditions calculated for bi-conformal vector fields are identically fulfilled in the neighbourhood of a point if and only if in such neighbourhood*

$$T_{abc} = 0, \qquad T^{d}{}_{cab} = 0,$$

whenever $P^{a}{}_{a}, \Pi^{a}{}_{a} \neq 3$.

5. Geometric characterization of bi-conformally flat pseudo-Riemannian manifolds

Once we have found the mathematical characterization of the spaces admitting a maximum number of bi-conformal vector fields we must next settle if there is actually any space whose metric tensor complies with the conditions stated in Theorem 9 or on the contrary there are no pseudo-Riemannian manifolds fulfilling such requirement. Indeed, we will find that each geometric condition has a separate meaning related to the geometric characterization of certain *separable* pseudo-Riemannian manifolds. Hence the tensors T_{abc} and $T^{a}{}_{bcd}$ bear a geometric interest on their own regardless of the existence of bi-conformal vector fields on the pseudo-Riemannian manifold where they are defined. Before addressing this issue we need some preliminary definitions.

Definition 10. The pseudo-Riemannian manifold (V, g_{ab}) is said to be conformally separable at the point $q \in V$ if there exists a local coordinate chart $x \equiv \{x^1, \ldots, x^n\}$ based at q in which the metric tensor takes the form

$$g_{ab}(x) = \begin{cases} \Xi_1(x)G_{\alpha\beta}(x^{\gamma}), & 1 \leq \alpha, \beta, \gamma \leq p, \\ \Xi_2(x)G_{AB}(x^{C}), & p+1 \leq A, B, C \leq n, \\ 0, & \text{otherwise.} \end{cases} \tag{68}$$

where Ξ_1, Ξ_2 are C^2 functions on the open set defining the coordinate chart. (V, g_{ab}) is conformally separable if it is so at every point $q \in V$. Any of the tensors $\Xi_1 G_{\alpha\beta}$, $\Xi_2 G_{AB}$ shall be called leaf metric.

In paper I we proved that a pseudo-Riemannian manifold is conformally separable at a point q if and only if the tensor T_{abc} is zero in a neighbourhood of q. In this case the orthogonal projectors P_{ab} and Π_{ab} used to construct T_{abc} generate integrable distributions D and D' (see considerations coming after Definition 1) and hence the set of integral manifolds of each distribution is a foliation in a neighbourhood of q. These foliations allow us to construct a local coordinate system around p in which the metric tensor takes the form (68) and the only non-vanishing components of the projectors are

$$P_{\alpha\beta} = \Xi_1(x)G_{\alpha\beta}(x^{\gamma}), \qquad \Pi_{AB} = \Xi_2(x)G_{AB}(x^{C}),$$

from which we conclude that P_{ab} and Π_{ab} are just the leaf metrics. The advantage of the characterization of conformally separable pseudo-Riemannian manifolds given in paper I is clear as it is invariant and coordinate-free. Therefore we conclude that any pseudo-Riemannian manifold fulfilling the conditions of Theorem 9 must be conformally separable.

In what is to follow we will work with conformally separable pseudo-Riemannian manifolds at a point and thus we will identify P_{ab}, Π_{ab} and the leaf metrics.

Bi-conformally flat spaces are a particular and interesting case of conformally separable pseudo-Riemannian manifolds defined by the requirement that the metrics $G_{\alpha\beta}$, G_{AB} be flat (so the leaf metrics are conformally flat). To our knowledge the study of bi-conformally flat spaces has never been tackled in the literature as opposed to many other conformally separable pseudo-Riemannian manifolds. Next, we fill up this gap proving a local characterization of bi-conformally flat pseudo-Riemannian manifolds along the same lines as in the conformally separable case.

Theorem 11. *A conformally separable pseudo-Riemannian manifold at a point q with leaf metrics of rank greater than 3 is bi-conformally flat at the same point if and only if the tensor $T^a{}_{bcd}$ constructed from its leaf metrics is identically zero in a neighbourhood of q.*

Proof. We choose the same coordinates and notation for our conformally separable space as in Definition 10 (from now on we will label tensor indexes with lowercase Greek and uppercase Latin letters according to the leaf of the separation they refer to). In the local coordinates of (68) the non-vanishing Christoffel symbols are

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2\Xi_1} G^{\alpha\rho}(\partial_\beta(\Xi_1 G_{\alpha\rho}) + \partial_\gamma(\Xi_1 G_{\rho\beta}) - \partial_\rho(\Xi_1 G_{\beta\gamma})),$$

$$\Gamma^A{}_{BC} = \frac{1}{2\Xi_1} G^{AD}(\partial_B(\Xi_1 G_{CD}) + \partial_C(\Xi_1 G_{DB}) - \partial_D(\Xi_1 G_{BC})),$$

$$\Gamma^\alpha_{\beta A} = \frac{1}{2\Xi_1}\delta^\alpha{}_\beta \partial_A \Xi_1, \qquad \Gamma^A{}_{B\alpha} = \frac{1}{2\Xi_2}\delta^A{}_B \partial_\alpha \Xi_2,$$

where

$$G^{\alpha\beta} G_{\beta\rho} = \delta^\alpha{}_\rho, \qquad G^{AC} G_{CB} = \delta^A{}_B,$$

and the tensors $G^{\alpha\beta}$ and G^{AB} are used to raise Greek and Latin indexes, respectively. The components of M_{abc}, E_a, W_a are (henceforth, the components not shown in an explicit tensor representation are understood to be zero)

$$M_{\alpha AB} = \partial_\alpha(\Xi_2 G_{AB}), \qquad M_{A\alpha\beta} = -\partial_A(\Xi_1 G_{\alpha\beta}), \qquad E_A = -\partial_A \log|\det(\Xi_1 G_{\alpha\beta})|,$$
$$W_\alpha = -\partial_\alpha \log|\det(\Xi_2 G_{AB})|. \tag{69}$$

Therefore we get for the components of the bi-conformal connection

$$\bar{\Gamma}^\alpha{}_{\beta\phi} = \frac{1}{2\Xi_1}(\delta^\alpha{}_\beta \partial_\phi \Xi_1 + \delta^\alpha{}_\phi \partial_\beta \Xi_1 - G^{\alpha\rho} G_{\beta\phi} \partial_\rho \Xi_1) + \Gamma^\alpha{}_{\beta\phi}(G),$$

$$\bar{\Gamma}^A{}_{BC} = \frac{1}{2\Xi_2}(\delta^A{}_B \partial_C \Xi_2 + \delta^A{}_C \partial_B \Xi_2 - G^{AR} G_{BC} \partial_R \Xi_2) + \Gamma^A{}_{BC}(G),$$

$$\bar{\Gamma}^\alpha{}_{\beta C} = \bar{\Gamma}^A{}_{B\phi} = 0, \tag{70}$$

where $\Gamma^\alpha{}_{\beta\phi}(G)$ and $\Gamma^A{}_{BC}(G)$ are the Christoffel symbols of the metrics $G_{\alpha\beta}$ and G_{AB}, respectively. Using this we deduce after some algebra

$$\bar{R}^\alpha{}_{\beta\phi\gamma} = R^\alpha{}_{\beta\phi\gamma}, \qquad \bar{R}^A{}_{BCD} = R^A{}_{BCD}, \qquad \bar{R}^\alpha{}_{\beta F\phi} = \partial_F \bar{\Gamma}^\alpha{}_{\phi\beta}, \qquad \bar{R}^A{}_{BF\phi} = -\partial_\phi \bar{\Gamma}^A{}_{FB},$$

$$L^0_{\alpha\beta} = 2R_{\alpha\beta} + \frac{R^\gamma{}_\gamma}{1-p} g_{\alpha\beta}, \qquad L^1_{AB} = 2R_{AB} + \frac{R^C{}_C}{1-n+p} g_{AB}, \qquad L^0_{\alpha A} = L^1_{A\alpha} = 0,$$

$$L^0{}_{A\alpha} = \frac{2(2-p)}{p}\partial_A \bar{\Gamma}^\rho{}_{\alpha\rho}, \qquad L^1_{\alpha A} = \frac{2(2-n+p)}{n-p}\partial_\alpha \bar{\Gamma}^R{}_{AR}, \tag{71}$$

where $R_{\alpha\beta}$, R_{AB}, $R^{\gamma}{}_{\gamma}$ and $R^{C}{}_{C}$ are the Ricci tensors and Ricci scalars of $R^{\alpha}{}_{\beta\phi\gamma}$ and $R^{A}{}_{BCD}$, respectively (these curvature tensors are calculated from the leaf metrics and not from their conformal counterparts $G_{\alpha\beta}$ and G_{AB}). From this and Eqs. (16) and (70) we get that the only non-vanishing components of the tensor $T^{a}{}_{bcd}$ are

$$T^{\alpha}{}_{\beta\phi\gamma} = 2C^{\alpha}{}_{\beta\phi\gamma}, \qquad T^{A}{}_{BCD} = 2C^{A}{}_{BCD}, \tag{72}$$

being $C^{\alpha}{}_{\beta\phi\gamma}$ and $C^{A}{}_{BCD}$ the Weyl tensors constructed from each leaf metric through the relations

$$C^{\alpha}{}_{\beta\phi\gamma} = R^{\alpha}{}_{\beta\phi\gamma} + \frac{1}{2-p}(g_{\beta[\gamma}L^{0}_{\phi]\rho}g^{\rho\alpha} + \delta^{\alpha}{}_{[\phi}L^{0}_{\gamma]\beta}),$$

$$C^{A}{}_{BCD} = R^{A}{}_{BCD} + \frac{1}{2-n+p}(g_{B[D}L^{1}_{C]E}g^{EA} + \delta^{A}{}_{[C}L^{1}_{D]B}).$$

Hence $T^{\alpha}{}_{\beta\phi\gamma}$ and $T^{A}{}_{BCD}$ are both zero if and only if the leaf metrics are both conformally flat which proves the theorem. □

Remark 12. Note that the tensors $\bar{R}^{\alpha}{}_{\alpha F\phi}$ and $\bar{R}^{A}{}_{AF\phi}$ (no summation) do not vanish in general which means that the bi-conformal connection does not stem from a metric tensor in this case.

From the calculations performed above it is clear that Theorem 11 can be generalized to conformally separable spaces in which only one of the leaf metrics is conformally flat. To that end, we define the tensor

$$T(P)^{a}{}_{bcd} \equiv P^{a}{}_{r}P^{q}{}_{b}P^{s}{}_{c}P^{t}{}_{d}T^{r}{}_{qst}. \tag{73}$$

Theorem 13. *Under the assumptions of Theorem 11 a leaf metric is conformally flat if and only if the tensor $T(P)^{a}{}_{bcd}$ calculated from the leaf metric P_{ab} is equal to zero.*

Proof. From the proof of Theorem 11 and (73) we deduce that for a conformally separable pseudo-Riemannian manifold the only non-vanishing components of $T(P)^{a}{}_{bcd}$ are

$$T(P)^{\alpha}{}_{\beta\gamma\phi} = T^{\alpha}{}_{\beta\gamma\phi} = 2C^{\alpha}{}_{\beta\gamma\phi},$$

so the vanishing of $T(P)^{a}{}_{bcd}$ implies that the Weyl tensor calculated from the corresponding leaf metric is zero as well. □

In the case of any of the leaf metrics being of rank 3 it is clear from the above that the corresponding tensor $T(P)^{a}{}_{bcd}$ will be zero as the Weyl tensor of any three-dimensional pseudo-Riemannian metric vanishes identically. Hence the results presented so far cannot be used to characterize conformally separable pseudo-Riemannian manifolds with conformally flat leaf metrics. This lacking is remedied in the next theorem.

Theorem 14. *A conformally separable pseudo-Riemannian manifold at a point q has a conformally flat leaf metric of rank 3 if and only if the condition*

$$\bar{\nabla}_{[a}L^{0}_{b]c} = 0, \tag{74}$$

holds in a neighbourhood of q where L^{0}_{ab} is calculated from the leaf metric P_{ab} by means of (9).

Proof. To show this we will rely on the notation and calculations performed in the proof of Theorem 11. If the manifold is conformally separable, then the components of the tensor $\bar{\nabla}_{a}L^{0}_{bc}$

are

$$\bar{\nabla}_\alpha L^0_{\beta\epsilon} = \nabla_\alpha L^0_{\beta\epsilon}, \qquad \bar{\nabla}_\alpha L^0_{B\epsilon} = \frac{2(2-p)}{p}(\partial^2_{\alpha B}\bar{\Gamma}^\rho{}_{\epsilon\rho} - \bar{\Gamma}^\delta{}_{\alpha\epsilon}\partial_B\bar{\Gamma}^\rho_{\delta\rho}), \qquad \bar{\nabla}_A L^0_{\beta\epsilon} = \partial_A L^0_{\beta\epsilon},$$

$$\bar{\nabla}_A L^0_{B\epsilon} = \partial_A L^0_{B\epsilon} - \bar{\Gamma}^C_{AB} L^0_{C\epsilon},$$

where ∇_α is the covariant derivative compatible with the metric $\Xi_1(x)G_{\alpha\beta}$. Trivially

$$\bar{\nabla}_{[\alpha} L^0_{\beta]\epsilon} = \nabla_{[\alpha} L^0_{\beta]\epsilon}, \tag{75}$$

here the tensor $\nabla_{[\alpha} L^0_{\beta]\epsilon}$ is the Schouten tensor of the three-metric $\Xi_1(x)G_{\alpha\beta}$ which vanish if and only if $G_{\alpha\beta}$ is flat. Therefore to finish the proof of this theorem we must show that all the remaining components of $\bar{\nabla}_{[a} L^0_{b]c}$ are zero. These are

$$\bar{\nabla}_{[A} L^0_{B]\epsilon} = \frac{2(2-p)}{p}\partial^2_{[AB]}\bar{\Gamma}^\rho{}_{\epsilon\rho},$$

$$2\bar{\nabla}_{[\alpha} L^0_{B]\epsilon} = \frac{2(2-p)}{p}(\partial^2_{\alpha B}\bar{\Gamma}^\rho{}_{\epsilon\rho} - \bar{\Gamma}^\delta{}_{\alpha\epsilon}\partial_B\bar{\Gamma}^\rho_{\delta\rho}) - \partial_B L^0_{\alpha\epsilon}.$$

Clearly the first expression is zero and the second one is worked out by replacing the connection coefficients by their expressions given in (70) and using the identity

$$L^0_{\alpha\beta} = L^0_{\alpha\beta}(G) + (2-p)(2\sigma_{\alpha\beta} + G_{\alpha\beta}(\partial\sigma)^2), \qquad \sigma = \frac{1}{2}\log|\Xi_1(x)|,$$

$$\sigma_{\alpha\beta} = \hat{\nabla}_\alpha\hat{\nabla}_\beta\sigma - \partial_\alpha\sigma\partial_\beta\sigma, \qquad (\partial\sigma)^2 = G^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma,$$

where $L^0_{\alpha\beta}(G)$ is calculated using the curvature tensors computed from $G_{\alpha\beta}$ and $\hat{\nabla}$ is the connection compatible with this metric. The sought result comes after some simple algebraic manipulations. □

All in all the geometric conditions proven in Theorems 11, 13 and 14 provide a set of equations which can be used to determine if a given pseudo-Riemannian manifold (V, g_{ab}) is conformally separable at a point with respect to a pair of leaf metrics P_{ab}, Π_{ab} and to decide if any of the leaf metrics is conformally flat. Note that all our characterizations are coordinate-free and in fact they can be used as intrinsic definitions of conformal separability and bi-conformal flatness at a point as opposed to Definition (10) which is coordinate-dependent. What is more one can study specific examples of pseudo-Riemannian manifolds in which tensors P_{ab} and Π_{ab} with the properties (5) are defined and check if the aforementioned conditions are fulfilled which is far more easier than trying to prove the existence of the coordinate system of (68) (see Example 16 for a practical application).

5.1. Bi-conformally flat spaces as spaces with a maximal number of bi-conformal vector fields

From the above it is clear that the conditions imposed by Theorem 9 are satisfied by a non-trivial set of pseudo-Riemannian manifolds. Thus we deduce that a bi-conformally flat pseudo-Riemannian manifold can be also characterized by the existence of a maximum number of bi-conformal vector fields. On the other hand it is straightforward to check (Proposition 6.1 of [3]) that for these spaces any conformal Killing vector of the leaf metrics is a bi-conformal vector field of the metric g_{ab}. As the number of conformal Killing vectors for each leaf metric is the biggest possible as well we get at once that for a bi-conformally flat space the total number of linearly independent bi-conformal vector fields is

$$\tfrac{1}{2}(p+1)(p+2)+\tfrac{1}{2}(n-p+1)(n-p+2), \qquad p, n-p \neq 2,$$

which is the upper bound N for the dimension of any finite-dimensional Lie algebra of bi-conformal vector fields. Summing up we obtain the following result.

Theorem 15. *A pseudo-Riemannian manifold possesses N linearly independent bi-conformal vector fields* ($P^a{}_a > 3$, $\Pi^a{}_a > n-3$) *if and only if it is bi-conformally flat.*

Bi-conformally flat spaces in which any of the leaf metrics has rank 3 also admit N linearly independent bi-conformal vector fields (in fact the complete integrability conditions are satisfied for these spaces as a result of Theorem 14). However, we do not know yet if there are spaces with N linearly independent bi-conformal vector fields with either of the projectors P_{ab} or Π_{ab} projecting on a three-dimensional vector space other than bi-conformally flat spaces. This is so because in such case the complete integrability conditions (59) may in principle be fulfilled by other conformally separable spaces not necessarily with conformally flat leaf metrics. The true extent of this assertion and the complete characterization of spaces with N linearly independent bi-conformal vector fields under these circumstances will be placed elsewhere.

6. Examples

We present here examples illustrating how our techniques work in practical cases. All the algebraic calculations can be performed with any of the computer algebra systems available today (the system used here was GRTensorII [14]).

Example 16. Consider the four-dimensional pseudo-Riemannian metric given by

$$\mathrm{d}s^2 = \Psi^2 \sin^2\theta(\mathrm{d}t+\mathrm{d}\phi)^2 - \alpha\,\mathrm{d}t^2 + B^2(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2),$$

where $-\infty < t < \infty$, $0 < r < \infty$, $0 < \theta < \pi$, $0 < \phi < 2\pi$ and

$$\Psi = \Psi(r,\theta), \qquad \Phi = \Phi(r,\theta), \qquad B = B(r,\theta), \qquad \alpha = \alpha(r,\theta).$$

Let us prove that this metric is conformally separable with respect to the foliation defined by the condition $t = \text{const}$. To that end we need to find out the orthogonal projector P_{ab} whose associated distribution D generates the above foliation. The distribution D is clearly defined by the set of vector fields $\{\partial/\partial r, \partial/\partial\theta, \partial/\partial\phi\}$ which are already orthogonal and hence Eq. (6) tells us how to construct P_{ab} and Π_{ab}. The only non-vanishing components of P_{ab} are

$$P_{tt} = \Psi^2\sin^2\theta, \qquad P_{rr} = B^2, \qquad P_{\theta\theta} = r^2B^2,$$
$$P_{\phi\phi} = \Psi^2\sin^2\theta, \qquad P_{t\phi} = \Psi^2\sin^2\theta,$$

and from this we can easily check that the tensor T_{abc} is zero identically. The tensor $\bar{\nabla}_{[a}L^0_{b]c}$ is not zero unless further restrictions are imposed. For instance if we set $\Psi^2 = r^2(1+r^2)$, $B^2 = 1+r^2$, $\alpha = -r^2$ the metric tensor becomes

$$\begin{aligned}\mathrm{d}s^2 &= r^2(1+(1+r^2)\sin^2\theta)\,\mathrm{d}t^2 + 2r^2(1+r^2)\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\phi\\ &\quad + (1+r^2)(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\,\sin^2\theta\,\mathrm{d}\phi^2),\end{aligned} \tag{76}$$

and a calculation shows that condition (74) holds. Hence according to Theorem 14 the metric is locally bi-conformally flat. Note that this is by no means evident in the coordinate system of (76) and so our method has a clear advantage. In this case we can go even further and find the explicit coordinates bringing (76) into the canonical form (68) which is

$$\mathrm{d}s^2 = (x^2 + y^2 + z^2)\,\mathrm{d}T^2 + (1 + x^2 + y^2 + z^2)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2),$$

being the coordinate change

$$T = t, \qquad x = r\sin\theta\cos(t+\phi), \qquad y = r\sin\theta\sin(t+\phi), \qquad z = r\cos\theta.$$

Example 17. In the foregoing results we have only concentrated on conformally separable pseudo-Riemannian manifolds but nothing was said about manifolds foliated by conformally flat hypersurfaces and not conformally separable. To illustrate this case let us consider the four-dimensional pseudo-Riemannian manifold V covered by a single coordinate chart $x = \{x^1, x^2, x^3, x^4\}$ and whose metric tensor is

$$\mathrm{d}s^2 = \Phi(x)[(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2] + 2\sum_{i=1}^{3}\beta_i(x)\,\mathrm{d}x^i\,\mathrm{d}x^4 + \Psi(x)(\mathrm{d}x^4)^2, \tag{77}$$

where $\Phi(x)$, $\beta_i(x)$, $\Psi(x)$ are functions at least C^3 in the whole manifold. Clearly the above line element represents the most general four-dimensional pseudo-Riemannian manifold admitting a local foliation[4] by three-dimensional conformally flat Riemannian hypersurfaces (given by the condition $x^4 =$ const). Now if we consider the integrable distribution associated to this foliation we may define an orthogonal projector P_{ab} by means of (6). To achieve this, we take the vector fields $\{\partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3\}$ which span the aforementioned distribution and construct an orthonormal set out of them (they are already orthogonal so it is enough with normalising). Applying (6) we easily get

$$P_{11} = P_{22} = P_{33} = \Phi(x), \qquad P_{i4} = \beta_i(x), \quad i = 1, 2, 3, \qquad P_{44} = \sum_{i=1}^{3}\frac{\beta_i^2(x)}{\Phi(x)}.$$

Using this we can check condition (74) using the projector P_{ab} to calculate L^0_{ab} and see what is obtained. The result is that the tensor $\bar{\nabla}_{[a}L^0_{b]c}$ does not vanish in this case although a calculation shows the important property

$$P^r{}_a P^s{}_b P^q{}_c \bar{\nabla}_{[r}L^0_{s]q} = 0. \tag{78}$$

Theorem 18. *A necessary condition that a four-dimensional pseudo-Riemannian manifold can be foliated locally by conformally flat Riemannian hypersurfaces is Eq.* (78).

This result suggests that it may well be possible to generalize the conditions of Theorem 14 to pseudo-Riemannian manifolds of arbitrary dimension which are not conformally separable replacing these conditions by (78). In fact this example can be generalized if we consider pseudo-Riemannian manifolds of higher dimension locally foliated by conformally flat hypersurfaces of dimension arbitrary and not necessarily Riemannian. The condition which must be checked in this case is $T(P)^a{}_{bcd} = 0$ being this condition found to be true in all the examples tried. Therefore it seems that a simple modification of the conditions of Theorems 13 and 14 holds even though the pseudo-Riemannian manifold is not conformally separable. The true extent of this assertion is under current research.

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[4] By local we mean a foliation defined in a neighbourhood of a point which is covered by a coordinate chart.

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